

Selected results on Lie supergroups and their radial operators

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Abstract

Foundational material on complex Lie supergroups and their radial operators is presented. In particular, Berezin's recursion formula for describing the radial parts of fundamental operators in general linear and ortho-symplectic cases is proved. Local versions of results which are suitable for applications for computing characters which are only defined on proper subdomains or covering spaces thereof are established.

Using Howe-duality in a Fock space context one observes that certain integrals of physical importance can be interpreted as numerical parts of characters of holomorphic semigroup representations of pieces of Lie supergroups which appear as Howe partners of classical symmetry ([CFZ, HPZ]). The characters are holomorphic superfunction of a parameter which is varying in a covering space of a domain in the base complex reductive group of the Lie supergroup at hand. Such covering spaces, which arise, e.g., due to the involvement of the metaplectic representation, can be regarded as the domains of definition of the semigroup representations. They contain pieces of maximal tori so that at least in a local sense one has the appropriate notions of radial functions and operators. In fact the restrictions χ of the character to this torus piece is the function, i.e., the integral, which is of interest.

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The character property implies that the integrals are eigenfunctions of the radial parts of Laplace-Casimir operators. In fact, in the cases considered in [CFZ, HPZ] the eigenvalues are zero! Thus in those cases every element D of the center universal enveloping algebra yields a differential equation which is of the form $\dot{D}(\chi) = 0$ where \dot{D} is the associated radial part of D .

The above is conceptually pleasing, but to complete the task of obtaining an explicit formula for the correlation functions χ one needs more concrete information on the radial parts \dot{D} of the Laplace-Casimir operators. Although all of the necessary information is contained in the fundamental work of F. Berezin, gleaning it from ([B]) and adapting it to the local setting indicated above requires a serious effort. Originally the first author of the present paper had hoped that it would be possible to provide at least a roadmap of [B] and include this in [HPZ]. However, this proved to be an unsatisfactory solution, in particular due to the additional length. Furthermore, we felt that more than a roadmap is needed. Thus our work developed into the thesis project [Ka1], the second author's paper [Ka2] and our work here.

Let us now briefly summarize this paper. The first section is primarily devoted to an explanation of the method of Grassmann analytic continuation and its application to defining a Lie supergroup associated to a given Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$. We do discuss Berezin's definitions of the morphisms of multiplication, inverse and the unit. However, since we are primarily interested in the differential operators, we pay most attention to the representations of \mathfrak{g} as left- and right-invariant derivations on the structure sheaf.

The remainder of the paper is devoted to considerations of radial operators, primarily in the cases of \mathfrak{gl} and \mathfrak{osp} which are of interest for the above mentioned Fock space applications. The first general goal is to describe the radial part of a Laplace-Casimir operator D by $\dot{D} = J^{-1}P_D J$ where P_D is a constant coefficient polynomial differential operator on the given maximal torus of the Lie group G associated to \mathfrak{g}_0 . A number of assumptions are needed for this, in particular that the function J should be an eigenfunction of the second order Laplace-Casimir operator which is defined in the usual way by an invariant nondegenerate supersymmetric bilinear form. This function appears as the square-root of the superdeterminant of a Jacobian of a coordinate chart which identifies a neighborhood \mathcal{A} of a regular point in a maximal

torus with a product $\mathcal{A} \times \mathcal{B}$ where \mathcal{B} is a subsupermanifold which is defined as a local orbit of the Lie supergroup by conjugation (see the Appendix).

These assumptions are satisfied in the two cases of interest mentioned above. Furthermore, in both cases there is an important infinite series $\{F_\ell\}$ of elements of the center of the universal enveloping algebra which defines a series of $\{D_\ell\}$ Laplace-Casimir operators for which the constant coefficient operators P_{D_ℓ} can be described via a certain recursive procedure. For example, in the case of \mathfrak{osp} , if we use the standard basis for the standard Cartan algebra, then the polynomials F_ℓ are defined by $F_\ell = \sum \varphi_i^{2\ell} + (-1)^\ell \sum \phi_j^{2\ell}$. Following Berezin we write $\dot{D}_\ell = J^{-1}T(F_\ell)J$. The goal is then to understand the map $F_\ell \mapsto T(F_\ell)$. Identifying the polynomials F_ℓ with the constant coefficient operators which they define, the main result of Berezin is that $T(F_\ell) = F_\ell + Q_\ell$ where Q_ℓ is a polynomial in $F_1, \dots, F_{\ell-1}$. The proof of this result is discussed here in substantial detail in §2.

The last paragraphs of §2 are devoted to local versions of the formula mentioned above. First, we show that the global results of Berezin apply to give the same results on the local product neighborhoods $\mathcal{A} \times \mathcal{B}$. These then lift to the covering spaces mentioned at the outset to give global results there by applying the identity principle.

Finally, in the appendix we outline the proof of $\dot{D} = J^{-1}P_D J$ on $\mathcal{A} \times \mathcal{B}$ which is given in detail in ([Ka1], Chapter 4) by methods which are similar to those used by Helgason in the classical case. Of course this is only valid under the same conditions as Berezin's global result.

1 Lie supergroup structure

Here we explain Berezin's construction of an analytic Lie supergroup associated to a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$. For this the structure sheaf \mathcal{F} is the sheaf of germs of holomorphic maps of G_0 to the Grassmann algebra $\wedge \mathfrak{g}_1^*$ which we write as $\mathcal{O}_{G_0} \otimes \wedge \mathfrak{g}_1^*$. Since we are primarily interested in invariant differential operators, we concentrate on construction of the representations of \mathfrak{g} as derivations on \mathcal{F} which correspond to left and right multiplication in the classical Lie group case.

1.1 Grassmann envelope

The first step for the construction is to consider the graded tensor product $\mathfrak{g} \otimes \Lambda$ with an arbitrary (finite-dimensional) Grassmann algebra. We assume throughout that Λ is isomorphic to the Grassmann algebra of an N -dimensional vector space with $N \geq n := \dim(\mathfrak{g}_1)$ and speak of Λ as being generated by N (odd) independent elements.

$$(\mathfrak{g} \otimes \Lambda)_0 = \mathfrak{g}_0 \otimes \Lambda_0 \oplus \mathfrak{g}_1 \otimes \Lambda_1$$

and

$$(\mathfrak{g} \otimes \Lambda)_1 = \mathfrak{g}_0 \otimes \Lambda_1 \oplus \mathfrak{g}_1 \otimes \Lambda_0.$$

For homogeneous elements $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \Lambda$ one defines

$$[\alpha X, \beta Y] := (-1)^{|X||\beta|} \alpha \beta [X, Y].$$

Extending by linearity, this defines a Lie superalgebra structure on the tensor product. Equipped with this structure, the *Grassmann envelope* $\mathfrak{g}(\Lambda) := (\mathfrak{g} \otimes \Lambda)_0$ is a usual Lie algebra. Note that in addition to being a Lie algebra, $\mathfrak{g}(\Lambda)$ is a Λ_0 -module. For notational convenience we write the multiplication by elements of Λ on the left, i.e., $X \otimes \alpha := (-1)^{|X||\alpha|} \alpha X$. Although we regard Λ as a variable, we suppress it notationally by letting $\tilde{\mathfrak{g}} := \mathfrak{g}(\Lambda)$.

Decomposition of $\tilde{\mathfrak{g}}$

The Lie algebra \mathfrak{g}_0 is a subalgebra of $\tilde{\mathfrak{g}}$ which has a complementary ideal \mathfrak{k} which is generated by the homogeneous elements αX where $\deg(\alpha) \geq 1$. We write $\tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes \mathfrak{k}$ as a semidirect sum. Let $\tilde{\mathfrak{g}}_0$ denote the Lie subalgebra of $\tilde{\mathfrak{g}}$ which is generated by the homogeneous elements αX where $|\alpha| = 0$, i.e., where α is even. It follows that $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \ltimes \mathfrak{k}_0$, where $\mathfrak{k}_0 := \mathfrak{k} \cap (\mathfrak{g}_0 \otimes \Lambda_0)$ is the subalgebra of consisting even elements in the nilpotent Lie algebra \mathfrak{k} . Observe that the linear subspace \mathfrak{k}_1 of odd elements of \mathfrak{k} is stabilized by $\tilde{\mathfrak{g}}_0$, i.e., $[\tilde{\mathfrak{g}}_0, \mathfrak{k}_1] \subset \mathfrak{k}_1$.

Decomposition at the group level

Let K be the simply connected Lie group associated to the Lie algebra \mathfrak{k} . Recall that $\exp : \mathfrak{k} \rightarrow K$ is a diffeomorphism. If we embed \mathfrak{k} as a Lie algebra of matrices, \exp is even polynomially defined with a polynomial inverse.

In particular it is bianalytic or biholomorphic, depending on the setting at hand. Let G_0 be a (connected) Lie group associated to \mathfrak{g}_0 which integrates the representation of \mathfrak{g}_0 on \mathfrak{g}_1 . It follows that G_0 acts on K and $\tilde{G} := G_0 \ltimes K$ is associated to $\tilde{\mathfrak{g}}$. Since G_0 acts on the Lie group $K_0 := \exp(\mathfrak{k}_0)$, the semidirect product $\tilde{G}_0 := G_0 \ltimes K_0$ is a closed Lie subgroup of \tilde{G} . The linear space \mathfrak{k}_1 of odd elements of \mathfrak{k} defines a submanifold $K_1 := \exp(\mathfrak{k}_1)$ of K which is stabilized by \tilde{G}_0 -conjugation. We write $\tilde{G} = \tilde{G}_0 K_1$ which is in a certain sense also a semidirect product.

1.2 Grassmann canonical coordinates

Here, starting with canonical coordinates on G_0 , we construct Grassmann canonical coordinates on \tilde{G} .

Canonical coordinates on G_0

Let U be an open neighborhood of $0 \in \mathfrak{g}_0$ so that $\exp : U \rightarrow G_0$ is a diffeomorphism onto an open neighborhood V of $\text{Id} \in G_0$. For $g \in G_0$ we have the open neighborhood $V.g$ which is identified with the coordinate chart U via the (bianalytic/biholomorphic) diffeomorphism $\exp : U \rightarrow V$. We shrink U to U_1 so that products of two elements as well as inverses of elements in $V_1 = \exp(U_1)$ are still contained in V and can still be identified with elements of U . To simplify notation, the possibly smaller sets U_1 and V_1 are still denoted by U and V , respectively. We cover G_0 by neighborhoods of this form which satisfy the further condition that if Vg_1 and Vg_2 have nonempty intersection, then $g_1 g_2^{-1} \in V$.

In order to obtain concrete coordinates we let $\{X_1, \dots, X_m\}$ be a basis of \mathfrak{g}_0 and to $X = \sum x_i X_i \in U$ associate the m -tuple $x = (x_1, \dots, x_m)$. If $g \in Vg_1 \cap Vg_2$ has coordinate x with respect to the chart Vg_1 and z with respect to Vg_2 , and $g_1 g_2^{-1} = \exp(X)$, then change of coordinates is computed by applying the Campbell-Baker-Hausdorff formula (CBH) to

$$\exp\left(\sum x_i X_i\right) \exp(X) = \exp\left(\sum z_i X_i\right).$$

Grassmann coordinates on \tilde{G}_0

Recall that $\exp : \mathfrak{k}_0 \rightarrow K_0$ is a diffeomorphism. Thus for U as above we can view $U \times \mathfrak{k}_0$ as a coordinate neighborhood of Id : Let $x = (x_1, \dots, x_m)$

be as above and $h = (h_1, \dots, h_m)$ be an m -tuple of elements $h_i \in \Lambda$. Then coordinates on $V \times K_0$ are given by the diffeomorphism $U \times \mathfrak{k}_0 \rightarrow VK_0 \cong V \times K_0$, $(x, h) \rightarrow \exp(\sum x_i X_i) \exp(\sum h_i X_i)$. Using the following remark, we will express these coordinates in a more convenient way.

Proposition 1.1. *The restriction of the \tilde{G}_0 -exponential map to $U \times \mathfrak{k}_0$ is a diffeomorphism onto its image VK_0 .*

Proof. For $(A, B) \in U \times \mathfrak{k}_0$ with B sufficiently small, i.e., sufficiently near $0 \in \mathfrak{k}_0$, we apply CBH to obtain a mapping $B \rightarrow \beta(A, B)$ defined by $\exp(A + B) = \exp(A) \exp(\beta(A, B))$. Now the definition of β involves brackets where B appears in a given (higher order) bracket at most N times, where Λ is the Grassmann algebra of a vector space of dimension N . If we use the coordinate h to describe B , this means that for fixed A the mapping β is polynomial in h . So we fix A and regard β as a map $\beta : \mathfrak{k}_0 \rightarrow \mathfrak{k}_0$. Analogously we define $\alpha : \mathfrak{k}_0 \rightarrow \mathfrak{k}_0$ by $\exp(A) \exp(C) = \exp(A + \alpha(C))$ in the range where CBH applies and then extend by the polynomial property. In the range where CBH applies we have $\exp(A + B) = \exp(A + \alpha\beta(B))$ and therefore $\alpha\beta(B) = B$ for B sufficiently small. Thus it follows from the polynomial property that $\alpha\beta = \text{Id}$. Arguing in the same way,

$$\exp(A) \exp(C) = \exp(A + \alpha(C)) = \exp(A) \exp(\beta\alpha(C))$$

implies that $\beta\alpha(C) = C$ for C sufficiently small and it follows from the polynomial property that $\beta\alpha = \text{Id}$. Hence $(A, B) \rightarrow \exp(A + B)$ is a bijective map $U \times \mathfrak{k}_0 \rightarrow VK_0$ and the desired result follows from its analyticity. In fact $\beta = \beta(A, B)$ is a polynomial map in B with coefficients analytic in A . \square

Now we cover G_0 with neighborhoods Vg as before and as a result have coordinates (x, h) on $VK_0g = VgK_0$ given by $(x, h) \rightarrow \exp(\sum (x_i + h_i) X_i)g$. Note that in all considerations the Grassmann variables $z_i = x_i + h_i$ behave as scalars.

Canonical coordinates on \tilde{G}

Now let $\{\Xi_1, \dots, \Xi_n\}$ be a basis of \mathfrak{g}_1 . An n -tuple $\xi = (\xi_1, \dots, \xi_n)$ with $\xi_j \in \Lambda_1$ defines $\sum \xi_j \Xi_j \in \mathfrak{k}_1$ and, allowing ξ to be arbitrary, we view it as a coordinate on \mathfrak{k}_1 . The exact same argument as that above shows that $(x, h, \xi) \rightarrow \exp(\sum (x_i + h_i) X_i + \sum \xi_j \Xi_j)$ defines a diffeomorphism (analytic/biholomorphic) $U \times \mathfrak{k} \rightarrow VK$. Again covering G_0 as above,

we cover \tilde{G} with neighborhoods VKg on which we have the coordinates $(x, h, \xi) \rightarrow \exp(\sum(x_i + h_i)X_i + \sum \xi_j \Xi_j)g$. These are the promised *Grassmann canonical coordinates*.

1.3 The \mathfrak{g} -representation on $\mathcal{O}_{\tilde{G}} \otimes \Lambda$.

For X a homogeneous element of \mathfrak{g} , let α be a homogeneous element of Λ with $|\alpha| = 1$ if $|X| = 0$ and $|\alpha| = 1$ if $|X| = 1$ so that in particular $|\alpha X| = 0$, and consider the 1-parameter subgroup $t \mapsto \exp(t\alpha X)$ in the Lie group \tilde{G} . For t sufficiently small we use CBH to express its action by left multiplication in Grassmann canonical coordinates:

$$\exp(t\alpha X)\exp(\sum(x_i + h_i)X_i + \sum \xi_j \Xi_j)g = \exp(\sum(x_i + h_i)X_i + \sum \xi_j \Xi_j + t\alpha\eta)g.$$

For this recall that the CBH computation that produces the righthand side of this equation involves higher order brackets where in theory αX can appear any number of times. However, using the definition of the Lie bracket, in particular that the Grassmann variables are behaving as scalars, along with the fact that $\alpha^2 = 0$, it follows that the only nonzero terms are those where αX appears exactly once.

Now, for homogeneous elements $Y, Z \in \mathfrak{g}_0$ and $\gamma, \delta \in \Lambda$, the bracket $[\gamma Y, \delta Z]$ is defined as $(-1)^{|Y||\delta|}\gamma\delta[Y, Z]$. But since $\tilde{\mathfrak{g}}$ is the even part of $\mathfrak{g} \otimes \Lambda$, it follows that $|\delta| = |Z|$. Thus, when transporting $t\alpha$ to the front of a bracket of homogeneous elements which occurs above, one only pays the price of a sign which does not depend on α . Therefore, for a fixed coordinate chart VKg , the element $\eta \in \mathfrak{g} \otimes \Lambda$ is well-defined independent of α .

Now let $\mathcal{L}_{\alpha X}$ be the Lie derivative defined by left multiplication by the 1-parameter group $\exp(t\alpha X)$.

Proposition 1.2. *There is an operator M_X on $\mathcal{O}_{G_0} \otimes \Lambda$ so that $\mathcal{L}_{\alpha X} = \alpha M_X$.*

Proof. Let the covering be denoted by $\{VKg_\ell\}$. As we just observed there exist operators M_X^ℓ on functions on VKg_ℓ which are defined independent of α so that $\mathcal{L}_{\alpha X}^\ell = \alpha M_X^\ell$. Here $\mathcal{L}_{\alpha X}^\ell$ denotes the expression of the globally defined Lie derivative in the local coordinates at hand. Regarding this as a vector field it obeys the change of coordinates rule $\mathcal{L}_{\alpha X}^k = J_{k\ell}\mathcal{L}_{\alpha X}^\ell$, where $J_{k\ell}$ is the Jacobian of the change of coordinates. Therefore $\alpha(M_X^k - J_{k\ell}M_X^\ell) = 0$ for every $\alpha \in \Lambda$. Now we embed Λ in a larger Grassmann algebra Λ' so that if $\beta \in \Lambda$ is such that $\alpha\beta = 0$ for all α in Λ' , then it follows that $\beta = 0$. We carry out the above construction for Λ' but,

with the exception of α , all computations are made with elements of $\tilde{\mathfrak{g}} = \mathfrak{g}(\Lambda)$. It follows that applying the difference $M_X^k - J_{k\ell}M_X^\ell$ to an element of $\mathcal{O}_{\tilde{G}} \otimes \Lambda$ results in a function with values $\beta \in \Lambda$ with $\alpha\beta = 0$ for all $\alpha \in \Lambda'$. Consequently, as desired $M_X^k = J_{k\ell}M_X^\ell$ as operators on $\mathcal{O}_{\tilde{G}} \otimes \Lambda$. \square

Representation theoretical properties of M

Define $M : \mathfrak{g} \rightarrow \text{End}(\mathcal{O}_{G_0} \otimes \Lambda)$ by $X \mapsto M_X$. Although the notation may not indicate it, M_X is an operator at the sheaf level. Observe that if X is homogeneous, then so is M_X and $|M_X| = |X|$. The main properties of M can be summarized as follows.

Proposition 1.3. *The mapping M is a representation of the Lie superalgebra \mathfrak{g} in the space $\text{Der}(\mathcal{O}_{\tilde{G}} \otimes \Lambda)$ of derivations of $\mathcal{O}_{\tilde{G}} \otimes \Lambda$.*

Proof. In the discussion α and β are chosen to be homogeneous elements in some larger Grassmann algebra Λ' with the property that $\alpha\beta\gamma = 0$ implies that $\gamma = 0$ for every $\gamma \in \Lambda$. For X, Y homogeneous elements in $\tilde{\mathfrak{g}} = \mathfrak{g}(\Lambda)$, we carry out the above construction for $\alpha X, \beta Y \in \mathfrak{g}(\Lambda')$. Since $|\alpha| = |X|$ and $|\beta| = |Y|$, it follows immediately from the definition of the Lie bracket that

$$\mathcal{L}_{[\alpha X, \beta Y]} = (-1)^{|M_X||M_Y|} \alpha\beta M_{[X, Y]}.$$

On the other hand,

$$\begin{aligned} \mathcal{L}_{[\alpha X, \beta Y]} &= \mathcal{L}_{\alpha X} \mathcal{L}_{\beta Y} - \mathcal{L}_{\beta X} \mathcal{L}_{\alpha Y} = \\ \beta \alpha M_X M_Y - \alpha \beta M_Y M_X &= \alpha \beta ((-1)^{|M_X||M_Y|} M_X M_Y - M_Y M_X) \end{aligned}$$

and therefore

$$\alpha \beta (M_{[X, Y]} - (M_X M_Y - (-1)^{|M_X||M_Y|} M_Y M_X)) = 0.$$

Applying this identity to $f \in \mathcal{O}_{\tilde{G}} \otimes \Lambda$ and using the cancellation property of $\alpha\beta$ yields the representation property of M .

In order to show that M_X is a derivation, we choose $\alpha \in \Lambda'$ with the cancellation property and note that for $f, g \in \mathcal{O}_{\tilde{G}} \otimes \Lambda$

$$\begin{aligned} \alpha M_X(fg) &= \mathcal{L}_{\alpha X}(fg) = \\ \mathcal{L}_{\alpha X}(f)g + f \mathcal{L}_{\beta X}g &= \alpha M_X(f)g + f \alpha M_X(g) = \\ \alpha(M_X(f)g + (-1)^{|f||X|} f M_X(g)) \end{aligned}$$

and the desired derivation property follows by cancellation. \square

Before closing this section, we should note that the same discussion as above with left multiplication by the 1-parameter groups $t \mapsto \exp(t\alpha X)$ being replaced by right multiplication leads to analogous representations of \mathfrak{g} in $\text{Der}(\mathcal{O}_{\tilde{G}} \otimes \Lambda)$. If it is necessary to differentiate between the representations defined by left multiplication and those defined by right multiplication, we denote the former by $X \mapsto L_X$ and the latter by $X \mapsto R_X$. Of course these representations commute in the graded sense, i.e., $L_X R_Y = (-1)^{|X||Y|} R_Y L_X$ and the representation which corresponds to conjugation in the classical Lie group case is given by $X \mapsto L_X + R_X$.

1.4 The \mathfrak{g} -representation on $\mathcal{O}_{G_0} \otimes \wedge \mathfrak{g}_1^*$

Above we have constructed \mathfrak{g} -representations on the function algebra of the Grassman envelope \tilde{G} . Here we turn to the main task of this section which is to construct the representations of \mathfrak{g} on the structure sheaf $\mathcal{F} = \mathcal{O}_{G_0} \otimes \Lambda$ which correspond to the representations by invariant vector fields of a Lie algebra on the structure sheaf of an associated Lie group. The key idea is to extend (analytic/holomorphic) functions from G_0 to functions on the special Grassmann envelope \tilde{G} where $\Lambda = \wedge \mathfrak{g}_1^*$ and apply the representations constructed above.

Grassmann analytic continuation

Let $\Lambda = \wedge \mathfrak{g}_1^*$ and consider a Λ -valued analytic or holomorphic function f on an open subset V of G_0 . We (analytically) continue f to a function $\Psi(f)$ on the open subset VK of the Grassmann envelope \tilde{G} . Since the construction does not depend on the nature of V , we suppress it in the discussion and only discuss functions defined on the full group G_0 .

As a first step we will (analytically) continue f to a function on \tilde{G}_0 . For this we cover G_0 as usual by open sets of the form Vg where V is an open neighborhood of Id with $\exp : U \rightarrow V$ a diffeomorphism. Letting $\{X_1, \dots, X_m\}$ be a basis of \mathfrak{g}_0 we have Grassmann coordinates on VK_0g given by

$$(x, h) \mapsto \exp\left(\sum (x_i + h_i)X_i\right)$$

on VK_0 and then composing by multiplication on VK_0g . Let g_k and g_ℓ be such that the coordinate neighborhoods Vg_k and Vg_ℓ have nonempty intersection. Since by assumption f is analytic (or holomorphic), in the respective coordinate charts it has convergent power series representation $f_k(x)$ and $f_\ell(x)$. Now a given point $v \in Vg_k \cap Vg_\ell$ is represented as

$$v = \exp\left(\sum x_i^k X_i\right)g_k = \exp\left(\sum x_i^\ell X_i\right)g_\ell.$$

with $g_\ell g_k^{-1} = \exp(X) \in V$. As we underlined above, if (x, h) are Grassmann coordinates defined by

$$(x, h) \mapsto \exp\left(\sum (x_i + h_i)X_i\right)g$$

where $g = g_k$ or $g = g_\ell$, and the change of variables at the level of G_0 , which is computed by CBH, is given by $x^k = A_{k\ell}(x^\ell)$, then the change of variables for sum $x + h$ is given by the same rule, i.e.. $x^k + h^k = A_{k\ell}(x^\ell + h^\ell)$.

Now, since the elements h of the Grassmann algebra are nilpotent, the power series $f_k(x + h)$ and $f_\ell(x + h)$ converge on $U \times \mathfrak{k}_0$, and since the transformation rule for the variable x is the same as that for $x + h$, the resulting locally defined functions agree on the intersection $VK_0g_k \cap VK_0g_\ell$. Thus we have defined the basic first step of *Grassmann analytic continuation* which is a continuous morphism of sheaves of algebras

$$GAC : \mathcal{O}_{G_0} \otimes \Lambda \rightarrow \mathcal{O}_{\tilde{G}_0} \otimes \Lambda.$$

The next step, i.e., the continuation to $\mathcal{O}_{\tilde{G}} \otimes \Lambda$, is formal. For this we observe that \tilde{G} is the product $\tilde{G}_0 \times \mathfrak{k}_1$. Using a basis $\{\Xi_1, \dots, \Xi_m\}$ for \mathfrak{g}_1 , an element of $\mathfrak{g}_1 \otimes \Lambda_1$ is written as $\sum \sigma_j \Xi_j$. Thus the product structure is given by $(\tilde{g}_0, \sigma) \mapsto \tilde{g}_0 \exp(\sum \sigma_j \Xi_j)$. But the dependence of $\exp(\sum \sigma_j \Xi_j)$ on the σ_j is polynomial. Consequently, if $\{\Xi_1^*, \dots, \Xi_m^*\}$ is the dual basis of \mathfrak{g}_1^* , then an arbitrary function $f \in \mathcal{O}_{\tilde{G}} \otimes \Lambda$ can be expressed as

$$f = \sum_{|I| \leq m} f_I \Xi_I^* = \sum_{i_1 < \dots < i_p} f_{i_1, \dots, i_p} \Xi_{i_1}^* \wedge \dots \wedge \Xi_{i_p}^*,$$

where the coefficient functions are arbitrary analytic/holomorphic functions on \tilde{G}_0 .

Using the above description, the continuation of a function from \tilde{G}_0 is formal. Recalling that $\Lambda := \wedge \mathfrak{g}_1^*$, a function $f \in \mathcal{O}_{\tilde{G}_0} \otimes \Lambda$ is already of the form

$$f = \sum_{|I| \leq m} f_I \Xi_I^*.$$

Its continuation $\hat{f} \in \mathcal{O}_{\tilde{G}} \otimes \Lambda$ is simply defined as

$$\hat{f} := \sum_{|I| \leq m} f_I \Xi_I^*,$$

re-interpreting the elements $\Xi_j \in \mathfrak{g}_1^*$ of the dual basis as odd coordinate functions on \tilde{G} . The Grassmann analytic continuation from $\mathcal{O}_{G_0} \otimes \Lambda$ is then defined by

$$\Psi : \mathcal{O}_{G_0} \otimes \Lambda \rightarrow \mathcal{O}_{\tilde{G}} \otimes \Lambda, \quad f \mapsto \widehat{GAC}(f).$$

The following summarizes the construction. Recall that for this $\Lambda := \wedge \mathfrak{g}_1^*$.

Proposition 1.4. *Grassmann analytic continuation $\Psi : \mathcal{O}_{G_0} \otimes \Lambda \rightarrow \mathcal{O}_{\tilde{G}} \otimes \Lambda$ is an algebra morphism which is an isomorphism onto its image \mathcal{A} . The algebra \mathcal{A} is the set of functions $F \in \mathcal{O}_{\tilde{G}} \otimes \Lambda$ of the form*

$$F = \sum_{|I| \leq m} f_I \Xi_I^*$$

where $f_I = \text{GAC}(h_I)$ for $h_I \in \mathcal{O}_{G_0} \otimes \Lambda$.

It should be remarked that a function $F = \sum f_I \Xi_I^*$ is in the image \mathcal{A} if and only if the expansions of the coefficients f_I in the local coordinates (x, h) are power series in $x + h$. In other words, from the point of view of the coefficient functions the Grassmann variables $x + h$ are scalars.

The representations of \mathfrak{g} on the structure sheaf

Recall that at the beginning we deemed the structure sheaf \mathcal{F} to be the sheaf of germs of holomorphic maps from G_0 to $\wedge \mathfrak{g}_1^*$ which we denote here by Λ . By definition the numerical functions are just the analytic/holomorphic functions \mathcal{O}_{G_0} with the projection $\text{num} : \mathcal{F} \rightarrow \mathcal{O}_{G_0}$ being the obvious one. Since \mathcal{O}_{G_0} is canonically embedded in \mathcal{F} with $\ker(\text{num})$ as complement, in the language of supergroups the structure is *split*.

We complete the task of constructing the representations of \mathfrak{g} on \mathcal{F} by noting the following.

Proposition 1.5. *The image \mathcal{A} of Grassmann analytic continuation is invariant under the representations L and R of \mathfrak{g} on $\mathcal{O}_{\tilde{G}} \otimes \Lambda$.*

Proof. We must show that if X is a homogeneous element of \mathfrak{g} and

$$F(\tilde{g}_0 \exp(\sum_j \sigma_j \Xi_j)) = \sum f_I \sigma_I$$

is in \mathcal{A} , then $L_X(F)$ and $R_X(F)$ are likewise in \mathcal{A} . Therefore we compute $\mathcal{L}_{\alpha X}(F)$ in Grassmann analytic coordinates. For example, in a neighborhood of Id we must use CBH to convert the product

$$\exp(t\alpha X) \exp(\sum z_i X_i) \exp(\sum \sigma_j \Xi_j)$$

to the form of the Grassmann coordinates. Here $z = x + h$ is the Grassmann variable. The result is complicated, but is of the form

$$\exp\left(\sum_i (t\alpha\tilde{z}_i(z, \sigma) + z_i)X_i\right)\exp\left(\sum_j (t\alpha\tilde{\sigma}_j(z, \sigma) + \sigma_j)\Xi_j\right).$$

The key point is that the coefficients $\tilde{z}_i(z, \sigma) = \sum_J c_i^J(z)\sigma_J$ are polynomials in σ with coefficients which are power series in z . Now recall that applying F to such an expression yields

$$\sum f_I \exp\left(\sum_i (t\alpha\tilde{z}_i(z, \sigma) + z_i)X_i\right)((t\alpha\tilde{\sigma}(z, \sigma) + \sigma)_I).$$

Using the power series representations of the f_I when differentiating one obtains polynomials in σ with coefficients which are power series in the Grassmann variables z . Of course factoring out α doesn't change this structure and it follows that the resulting function is of the desired form, i.e.,

$$\mathcal{L}_{\alpha X}(F) = \alpha \cdot \sum \tilde{f}_I \sum \Xi_I^*$$

where the coefficients f_I are power series in Grassmann analytic coordinates. Note that although we have only verified this in the coordinates at the identity, the verification in a general neighborhood VK_1g only differs from that at the identity by conjugation of K_1 by $g \in G_0$. \square

As a consequence we have the main result of this section.

Theorem 1.6. *Restricting the representations L and R of \mathfrak{g} on $\mathcal{O}_{\tilde{G}} \otimes \wedge \mathfrak{g}_1^*$ to the image \mathcal{A} of Grassmann analytic continuation $\Psi : \mathcal{F} \xrightarrow{\cong} \mathcal{A}$ defines the representations of \mathfrak{g} on the structure sheaf \mathcal{F} . Restricted to \mathfrak{g}_0 these are given by differentiating elements of \mathcal{F} by the natural action of 1-parameter subgroups of G_0 on \mathcal{F} .*

1.5 Comments on Lie supergroups

A Lie supergroup is more than a structure sheaf \mathcal{F} with representations L and R which extend the natural representations of \mathfrak{g}_0 . The appropriate additional structure can be formulated in terms of a triple $(\mu, \iota, \varepsilon)$ of maps at the sheaf level which correspond to multiplication, inverse and the identity. For example, in the classical case the multiplication map $\mu : \mathcal{O}_G \rightarrow \mathcal{O}_{G \times G}$ is given by $\mu(f)(g_1, g_2) = f(g_1 g_2)$. Certain compatibility conditions must be fulfilled (see [B], p.247). Once this triple is defined, the representations L and R can be computed. Using Grassmann analytic continuation as above, Berezin does indeed define such a triple and then constructs L and R as above.

Berezin's construction of the Lie supergroup triple uses a notion of a superfunction which is slightly different from that above. In simple terms this is the difference between the notion of a variable and the evaluation of the variable. For example, a complex polynomial P in several commuting variables z_1, \dots, z_m is an element of the ring $\mathbb{C}[z_1, \dots, z_m]$. If R is any commutative \mathbb{C} -algebra, e.g., of numbers, functions, operators, etc., P can be evaluated to define a function $P_R : R \times \dots \times R \rightarrow R$. In the setting at hand ξ_1, \dots, ξ_n are anticommuting variables and Berezin defines a superfunction on G_0 to be an element $f \in \mathcal{O}_{G_0}[\xi_1, \dots, \xi_m]$. This is a function $f = f(g, \xi)$ in two blocks of variables. The first variables g can be regarded as the commuting variables defined by the coordinates in the manifold G_0 , e.g., in some power series ring. The second variables are the anticommuting variables which generate a Grassmann algebra $\Lambda(\xi_1, \dots, \xi_m)$.

Continuing with the simple example, if a_1, \dots, a_n are algebraically independent elements of a commutative ring R , then any algebraic combination $\sum c_I a^I$ defines a unique polynomial $P \in R[z_1, \dots, z_n]$. In our context where $n := \dim(\mathfrak{g}_1)$, one considers $\mathcal{O}_{G_0} \otimes \Lambda$ where Λ is a Grassmann algebra generated by N elements with $N \geq m$. If a_1, \dots, a_m are independent elements of Λ , then there is a canonical isomorphism between the subalgebra spanned by these elements and the algebra of superfunctions $\mathcal{O}_{G_0}[\xi_1, \dots, \xi_m]$.

As stated at the outset we wish to regard the structure sheaf \mathcal{F} as the sheaf of germs of holomorphic functions on G_0 with values in $\wedge \mathfrak{g}_1^*$. This may seem different from Berezin's sheaf, but if one exchanges variables for evaluated variables, it is not. The main point in this regard is that Berezin's construction uses a basis $\{\Xi_1, \dots, \Xi_n\}$ for \mathfrak{g}_1 . Thus evaluating a Berezin superfunction $f = f(g, \xi) \in \mathcal{O}_{G_0}[\xi_1, \dots, \xi_n]$ on Ξ_1^*, \dots, Ξ_n^* defines such a function. Conversely, starting with a holomorphic function $f : G_0 \rightarrow \wedge \mathfrak{g}_1^*$ and the basis $\{\Xi_1, \dots, \Xi_n\}$ one obtains a Berezin superfunction by replacing the Ξ_j^* by the variables ξ_j .

When defining the Lie supergroup triple $(\mu, \iota, \varepsilon)$ the variable viewpoint is very useful. For example, the standard multiplication morphism $\mu : G_0 \rightarrow G_0 \times G_0$ must be lifted to a map $\varphi : \mathcal{F}_{G_0} \rightarrow \mathcal{F}_{G_0 \times G_0}$ of sheaves. This map can be defined as follows. We regard a function in \mathcal{F}_{G_0} as a function of one variable (g, ξ) . Here g is already Grassmann continued to \tilde{G}_0 where Λ is generated by N odd elements with $N \geq 2n$. As usual ξ denotes an n -tuple of independent Grassmann variables. The functions in $\mathcal{F}_{G_0 \times G_0}$ are functions of two such variables $((g_1, \xi_1), (g_2, \xi_2))$. Now we *evaluate* these variables at points in the group \tilde{G} by using $2n$ independent odd elements $a_1, \dots, a_n, a_{n+1}, \dots, a_{2n}$ in Λ . The pair of variables defines a pair of group elements

$$g_1 \exp(\sum a_j \Xi_j), g_2 \exp(\sum a_{n+j} \Xi_j) \in \tilde{G}.$$

Using the decomposition $\tilde{G} = \tilde{G}_0 K_1$, the product of these group elements is written as $g \exp(\sum \eta_j \Xi_j)$. Now we change back to the variable standpoint and regard g as a Grassmann analytically continued variable in \tilde{G}_0 and $\eta = (\eta_1, \dots, \eta_n)$ as a variable which (as can be checked) consists of n independent odd Grassmann variables. Since the construction of manipulating the product to the form $g \exp(\sum_j \eta_j \Xi_j)$ only uses the *variable properties* of the elements a_j , the *product variable* (g, η) does not depend on the choice of the a_j . The multiplication morphism at the sheaf level is then defined in the natural way:

$$\varphi(f)((g_1, \xi_1), (g_2, \xi_2)) := f(g, \eta).$$

Lifting the inverse mapping $g \mapsto g^{-1}$ to the sheaf level is defined analogously. The evaluation at the identity is given by $\varepsilon(f)(g, \xi) = f(\text{Id}, 0)$. Since these operations are defined by the group structure of \tilde{G} , it can be directly checked that they have the desired compatibility properties. As indicated at the outset of this paragraph, if we regard \mathcal{F} as the sheaf of $\wedge \mathfrak{g}_1^*$ -valued holomorphic functions, then the Lie supergroup structure is defined at that level by replacing a function $\sum f_I(g)(\Xi^*)^I$ by a Berezin function $f = f(g, \xi)$ of two variables defined by $f(g, \xi) = \sum_I f_I(g) \xi^I$. Multiplication is then defined as above at the level of variables and then one returns to $\wedge \mathfrak{g}_1^*$ -valued holomorphic functions by evaluating the variables at Ξ_1^*, \dots, Ξ_n^* .

The representations L and R of \mathfrak{g} on \mathcal{F} are in fact by derivations which are invariant under the group structure. For example, invariance by right-multiplication means that

$$(M_X \otimes \text{Id}) \circ \varphi = \varphi \circ M_X \tag{1}$$

for $X \in \mathfrak{g}$. This is immediate for the derivations defined by even elements, because the supergroup multiplication by elements of \tilde{G}_0 is just usual multiplication. For odd elements $X \in \mathfrak{g}_1$ one must check that the formula analogous to 1 with M_X replaced by the Lie derivative $\mathcal{L}_{\alpha X}$ in \tilde{G} makes sense and is valid. Then one cancels α in the same way that was done for the definition of M_X .

There are various other methods of constructing a Lie supergroup associated to \mathfrak{g} , (see [K] and [Ka2] for constructions using Lie-Hopf algebras and [Ko] for the dual construction at the level of the structure sheaf). But it turns out that, even in the more delicate holomorphic setting, these are all equivalent (see [V]). For further discussion of this matter, e.g., for a detailed comparison of the various definitions, see [Ka1].

2 Radial operators

Our concrete goal here is to describe certain series of radial differential operators in sufficiently concrete form for computational applications in [CFZ] and [HPZ]. As in the previous section, the main results were originally proved by Berezin.

2.1 Basic definitions

Here we recall the basic objects for the study of radial differential operators on Lie supergroups.

Universal enveloping algebra

The *universal enveloping algebra* of a Lie superalgebra \mathfrak{g} is the quotient $U(\mathfrak{g})$ of the full tensor algebra $T(\mathfrak{g})$ by the ideal generated by $(X \otimes Y - (-1)^{|X||Y|} Y \otimes X) - [X, Y]$ where X, Y are homogeneous elements of \mathfrak{g} regarded as elements of (\mathfrak{g}) . If $X_1 \otimes X_2 \cdots \otimes X_k$ is a monomial in $T(\mathfrak{g})$, then its image in $U(\mathfrak{g})$ is equipped with the sign $(-1)^{|X_1| + \dots + |X_k|}$. This defines a \mathbb{Z}_2 -grading on $U(\mathfrak{g})$ for which the induced bracket defines a Lie superalgebra structure. Here we shall primarily be concerned with the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. This is the subalgebra of $U(\mathfrak{g})$ consisting of those elements X with $\text{ad}(X) = [X, \cdot] = 0$.

Laplace-Casimir operators

Recall that a representation of a Lie superalgebra \mathfrak{g} is a superalgebra morphism $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ to the superalgebra of linear maps of a graded vector space $V = V_0 \oplus V_1$. Such a representation extends to a representation $\rho : U(\mathfrak{g}) \rightarrow \text{End}(V)$. The examples of main importance here are the representations L and R of \mathfrak{g} constructed in the previous section on the structure sheaf \mathcal{F} of superfunctions of the associated Lie supergroup. In particular, we consider the representation $X \mapsto L_X$ and extend it to a representation of $U(\mathfrak{g})$ by differential operators. The *Laplace-Casimir operators* are those in the image of the center $Z(\mathfrak{g})$.

2.2 Radial functions

A superfunction f on G_0 is said to be *radial* if it is annihilated by the representation $X \mapsto L_X + R_X$ of \mathfrak{g} . Let \mathcal{R}_{G_0} denote the sheaf of radial holomorphic functions on G_0 . Observe that if D is a Laplace-Casimir operator, then $D|\mathcal{R}_{G_0} : \mathcal{R}_{G_0} \rightarrow \mathcal{R}_{G_0}$. We regard \mathcal{R}_{G_0} as the natural domain of definition of these operators.

Note that if f is globally defined on G_0 , then the condition $L_X f + R_X f = 0$ for all $X \in \mathfrak{g}_0$ just means that f is conjugation invariant, i.e., $f(g_0 g g_0^{-1}) = f(g)$ for all $g_0 \in G_0$. If G_0 is reductive, which we assume from now on, it follows that globally defined conjugation invariant superfunctions are completely determined by their restrictions to any given maximal torus H . We fix such a maximal torus and let \mathfrak{h} denote its Lie algebra in \mathfrak{g}_0 . Let us also assume, as will be the case in all applications, that \mathfrak{h} is a Cartan algebra of the full Lie superalgebra \mathfrak{g} .

Restriction theorem at the Lie superalgebra level

At the infinitesimal level we are interested in understanding the image of the restriction map from the space $S(\mathfrak{g}^*)^{\mathfrak{g}}$ of $\text{ad}_{\mathfrak{g}}$ -invariant (super) polynomials on \mathfrak{g} to the space $S(\mathfrak{h}^*)$. We view an element $P \in S(\mathfrak{g}^*)$ as a (holomorphic) polynomial map $P : \mathfrak{g}_0 \rightarrow \wedge \mathfrak{g}_1^*$. If U is a sufficiently small neighborhood of $0 \in \mathfrak{g}_0$ which is identified by the exponential map with a neighborhood V of Id in G_0 , then $P \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ if and only if the resulting function on V is radial.

Since it has been assumed that \mathfrak{h} is an even subspace of \mathfrak{g} , polynomials in $S(\mathfrak{h}^*)$ are just standard (numerically valued) polynomials. Since $P \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ is invariant by the adjoint representation of G_0 , it is immediate that $R(P)$ is invariant under the Weyl group $W = W(\mathfrak{g}_0, \mathfrak{h}_0)$. Thus we regard R as a map $R : S(\mathfrak{g}^*)^{\mathfrak{g}} \rightarrow S(\mathfrak{h}^*)^W$.

A great deal is known about the restriction morphism R . In particular, it is always injective. In our cases of interest, the basic results are proved in [B] and [S]. Let us quote Berezin's Theorem 3.1.

Theorem 2.1. *Let G_0 be reductive and assume that \mathfrak{h} is a Cartan algebra of \mathfrak{g} which is contained in \mathfrak{g}_0 . Assume further that \mathfrak{g} is endowed with a nondegenerate invariant scalar product and that its odd root spaces are 1-dimensional. Then a W -invariant polynomial is in the image of R if and only if for every odd root β (with dual root β^*) it follows that*

$$\left. \frac{d}{dt} \right|_{t=0} P(h + t\beta^*) = \beta(h)Q(h) \quad (2)$$

where $Q = Q(h)$ is a polynomial on \mathfrak{h} .

We say that the *extendible polynomials* are exactly those which are W -invariant and satisfy the *divisibility condition* (2). Since our work here is aimed at understanding properties of radial functions and operators in the cases of \mathfrak{gl} and \mathfrak{osp} , it should be emphasized that the conditions of Berezin's theorem are fulfilled in those cases.

For the statement of the version of Berizin's extension theorem for holomorphic functions we say that a holomorphic superfunction on \mathfrak{g}_0 is radial if and only if it is annihilated by the all $\text{ad}_{\mathfrak{g}}$ -derivations. The divisibility condition for holomorphic functions is the same as that for polynomials.

Theorem 2.2. *Under the assumptions of Theorem 2.1 it follows that a W -invariant function $f \in \mathcal{O}(\mathfrak{h})$ can be (uniquely) extended to a radial holomorphic superfunction on \mathfrak{g}_0 if and only if it satisfies the divisibility condition.*

Sketch of Proof. For the necessity, i.e., that the divisibility condition is really needed for extension, one replaces the polynomial P in Berezin's proof by the convergent power series representation of the given function f at $0 \in \mathfrak{h}$. For the sufficiency Berezin uses the fact that the given polynomial P can be extended to a unique Ad_{G_0} -invariant polynomial on \mathfrak{g}_0 and then proceeds by using generalities on superfunctions. Since W -invariant holomorphic functions on \mathfrak{h} extend to Ad_{G_0} -invariant holomorphic functions on \mathfrak{g}_0 , the same proof can be carried out in the holomorphic case. \square

Restriction theorem at the group level

Now let us turn to the Lie supergroup associated to \mathfrak{g} equipped with its sheaf \mathcal{F} of holomorphic superfunctions, i.e., the sheaf of germs of holomorphic maps with values in $\wedge \mathfrak{g}_1^*$. We assume that \mathfrak{g} satisfies the assumptions of Theorem 2.1 and let H be the maximal (complex) torus in G_0 associated to \mathfrak{h} . In this situation we wish to determine, e.g., the image of the restriction map $R : \mathcal{R}_{G_0}(G_0) \rightarrow \mathcal{O}(H)^W$ from the globally defined holomorphic radial functions on G_0 in the algebra of W -invariant holomorphic functions on H .

For this the divisibility condition must be transferred to the group level: A holomorphic function $f \in \mathcal{O}(H)$ is said to satisfy the divisibility condition if and only if its pull-back $f \circ \exp$ satisfies the divisibility condition on \mathfrak{h} . The following is an immediate consequence of the results in the previous paragraph.

Proposition 2.3. *A holomorphic function $f \in \mathcal{O}(H)^W$ satisfies the divisibility condition if and only if its lift $f \circ \exp$ is the restriction of a uniquely determined radial holomorphic superfunction on \mathfrak{g} .*

The extension theorem at the group level is stated as expected.

Theorem 2.4. *Let \mathfrak{g} be a Lie superalgebra which satisfies the conditions of Theorem 2.1 and let G_0 be a base of an associated Lie supergroup. Fix a Cartan algebra \mathfrak{h} in \mathfrak{g} and let $H = \exp(\mathfrak{h})$. Then a W -invariant holomorphic function $f \in \mathcal{O}(H)$ is the restriction of a radial holomorphic superfunction on G_0 if and only if it satisfies the divisibility condition.*

Proof. To prove the sufficiency of the divisibility condition we let f be a W -invariant (numerical) holomorphic function on H which satisfies the divisibility condition and $\hat{f} = f \circ \exp$. Since \hat{f} satisfies the divisibility condition on \mathfrak{h} , it is the restriction of a (unique) holomorphic radial superfunction $E(\hat{f})$ on \mathfrak{g}_0 . This means that \hat{f} is the numerical part of $E(\hat{f})|_{\mathfrak{h}}$. The uniqueness of the extension $E(\hat{f})$ implies that $E(\hat{f})|_{\mathfrak{h}}$ is periodic with respect to the discrete additive subgroup of \mathfrak{h} which is the kernel of $\exp : \mathfrak{h} \rightarrow H$. Thus there is a $\wedge \mathfrak{g}_1^*$ -valued holomorphic function f_s on H with $E(\hat{f})|_{\mathfrak{h}} = f_s \circ \exp$.

Observe that since $E(\hat{f})$ is invariant by conjugation by elements of the normalizer of H , it follows that its restriction to \mathfrak{h} is W -invariant and thus f_s is W -invariant. Classical invariant theory then implies that f_s is the restriction of a unique conjugation invariant superfunction $E(f)$ on G_0 . The function $E(f) \circ \exp$ is an Ad_{G_0} -invariant holomorphic superfunction which agrees with $E(\hat{f})$ on \mathfrak{h} . Thus $E(f) \circ \exp$ is the radial extension of \hat{f} . In particular, if U is a neighborhood of $0 \in \mathfrak{g}_0$ such that $\exp : U \rightarrow V$ is biholomorphic, then the fact that the operators $L_X + R_X$ annihilate $E(\hat{f})|_U$ for all $X \in \mathfrak{g}$ implies that they annihilate $E(f)|_V$ for all $X \in \mathfrak{g}$. Hence the identity principle implies that $(L_X + R_X)E(f) = 0$ for all $X \in \mathfrak{g}$ and consequently $E(f)$ is the desired radial extension of f .

For the necessity of the divisibility condition, we just reverse the argument: If $E(f)$ is the radial extension of f , then $E(f) \circ \exp|_U$ is annihilated by the operators $L_X + R_X$ and the identity principle implies that $E(f) \circ \exp$ is the radial extension of \hat{f} . Consequently \hat{f} satisfies the divisibility condition which by definition is the divisibility condition for f . \square

2.3 Jacobian formula

Recall that we have regarded the Laplace-Casimir operators as being differential operators $D : \mathcal{R} \rightarrow \mathcal{R}$ on the sheaf of radial holomorphic superfunctions on the Lie supergroup associated to a Lie superalgebra \mathfrak{g} . Here we restrict our considerations to the setting of Theorem 2.1 so that the algebra of global radial functions is described by the divisibility condition along a given maximal torus H in G_0 . Note that divisibility at the group level means that for every odd root β the directional derivative $X_\beta(f)$ is divisible by $(r_\beta - 1)$ where r_β is the character associated to β .

Denote by \mathcal{D}_H the image in $\mathcal{O}(H)^W$ of the restriction map R and $E : \mathcal{D}_H \rightarrow \mathcal{R}(G_0)$ the extension map which is R^{-1} . The associated *radial part* of a Laplace-Casimir operator D on H is defined as $\dot{D} := RDE : \mathcal{D}_H \rightarrow \mathcal{D}_H$. Our goal in this paragraph is to describe a basic result of Berezin which, at least for the Lie superalgebras and operators of main interest for the applications in [CFZ] and [HPZ], shows that

the study of the radial operators \dot{D} can be reduced to analyzing certain constant coefficient polynomial differential operators on H . This is proved by introducing a sort of change of variables along H so that generically along H one has a local product decomposition in the H -direction and the transversal direction of the supergroup action. As a consequence, a Jacobian J appears and therefore we refer to the result as the Jacobian formula.

The Jacobian J is defined as follows as a meromorphic function on H : For $\zeta = \exp(t) \in H$

$$J(\zeta) := \frac{\prod_{\alpha \in \Delta_0^+} 2 \sin \frac{\alpha(t)}{2}}{\prod_{\beta \in \Delta_1^+} 2 \sin \frac{\beta(t)}{2}}$$

where $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$ is a system of even and odd positive roots. Under certain restrictive conditions the Jacobian formula states that given a Laplace-Casimir operator D there is a uniquely defined polynomial operator with constant coefficients P_D on H so that $\dot{D} = J^{-1}P_D J$. It should be remarked that in the classical case of Lie groups the analogous formula (without the odd roots in the Jacobian) holds in great generality. In the Lie supergroup setting we state it in the cases of $\mathfrak{g} = \mathfrak{gl}(m, n)$ and $\mathfrak{osp}(2m, 2n)$. In [B] the latter Lie superalgebra is denoted by $C(m, n)$. From now on \mathfrak{g} is restricted to be one of these (complex) Lie superalgebras equipped with the nongenerate bilinear form $(X, Y) = \text{STr}(XY)$.

In the case of $\mathfrak{gl}(m, n)$ we choose \mathfrak{h} to be the Cartan algebra of diagonal matrices with coordinates

$$h = \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}.$$

One checks that the polynomial function

$$F_k = \text{STr}(h^k) = \sum \varphi_i^k - \sum \psi_j^k$$

satisfies the divisibility condition and therefore is extendible to an ad-invariant element of $S(\mathfrak{g}^*)$, i.e., to an element of $Z(\mathfrak{g})$.

In the case of $\mathfrak{osp}(2m, 2n)$ we recall that $\mathfrak{g}_0 = \mathfrak{so}_{2m} \oplus \mathfrak{sp}_{2n}$ and as above choose \mathfrak{h} to be in diagonal form with coordiantes

$$h_{\mathfrak{so}} = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix}$$

and

$$h_{\mathfrak{sp}} = \begin{pmatrix} \psi & 0 \\ 0 & -\psi \end{pmatrix}$$

with the full Cartan algebra given by $h = \text{Diag}(h_{\mathfrak{so}}, h_{\mathfrak{sp}})$. In this case one defines the extendible polynomials

$$F_k := \frac{1}{2} \text{STr}(h^{2k}) = \sum \varphi_i^{2k} - \sum \psi_j^{2k}.$$

In the case of $\mathfrak{gl}(m, n)$ the algebra of extendable polynomials consists of polynomials with constant coefficients in the F_k . In the case of $\mathfrak{osp}(2m, 2n)$ one requires one additional generator which is most conveniently chosen as $L = \varphi_1 \cdots \varphi_m R$ where R is the product of the odd positive root functions. For the sake of brevity of notation we let

$$\tilde{F}_k = F_k \left(\frac{1}{i} \frac{\partial}{\partial t} \right)$$

be the constant coefficient differential operator defined by F_k . A simplified version of Berezin's Jacobian theorem (see Theorem 3.2 on p. 302 of [B]) can be stated as follows.

Theorem 2.5. *Let \mathfrak{g} be either $\mathfrak{gl}(m, n)$ or $\mathfrak{osp}(2m, 2n)$ and P be an extendible polynomial on \mathfrak{h} which defines the radial differential operator \dot{D}_P by $\dot{D}_P(f) = R D_P E(f)$ for $f \in \mathcal{D}_H$ a W -invariant function satisfying the divisibility condition. Then there exists a uniquely determined polynomial function $T(P)$ on \mathfrak{h} with associated constant coefficient differential operator denoted by $T(P)(\frac{1}{i} \frac{\partial}{\partial t})$ so that*

$$\dot{D}_P = J^{-1} T(P) \left(\frac{1}{i} \frac{\partial}{\partial t} \right) J.$$

Furthermore, if $P = F_k$, then $T(P)$ is of degree k with top degree term F_k .

It should be underlined that the partial derivative operators

$$\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_{m+n}} \right)$$

are defined by the coordinates (φ, ψ) of \mathfrak{h} which were introduced above.

Remark. For applications in [HPZ] the first author was originally interested in a local version of Theorem 2.5 at a generic point of a maximal torus. Using techniques which are much closer to the methods of Helgason in the classical case, the second author has proved such a result in his thesis [Ka1]. As is shown in the

sequel, Berezin's global result Theorem 2.5 implies the local result. Vice versa, application of the identity principle shows that the local result implies Berezin's global result. Thus the two results are equivalent. Nevertheless we feel that it is of interest to have new viewpoint on these matters. Hence, the statement and a sketch proof of the local theorem have been included as an appendix to this paper.

Now we turn to understanding the mapping $P \mapsto T(P)$. This will be discussed for the polynomials $P = F_k$ in both cases $\mathfrak{gl}(m, n)$ and $\mathfrak{osp}(2m, 2n)$.

2.4 Finite-dimensional representations

Here we explain how to compute the polynomials $T(P)$ in terms of the eigenvalues of the radial operators \dot{D}_P on characters of *finite-dimensional* representations. We restrict to the cases $\mathfrak{g} = \mathfrak{gl}(m, n), \mathfrak{osp}(2m, 2n)$, but most of the discussion applies in a much more general setting, e.g., where \mathfrak{g}_0 is semisimple. For more details see Chapter 3.10 (p.307-311) of [B].

Character formula

Consider a finite-dimensional irreducible representation ρ of a complex Lie supergroup associated to one of the Lie superalgebras $\mathfrak{g} = \mathfrak{gl}(m, n), \mathfrak{osp}(2m, 2n)$. This is by definition a homomorphism of the Lie supergroup associated to \mathfrak{g} to that associated to Lie superalgebra $\mathfrak{gl}(V)$. Such is defined by a holomorphic mapping $G_0 \rightarrow \mathrm{GL}(V)$ which lifts to the sheaf level as a mapping $\mathcal{F}_{\mathrm{GL}(V)} \rightarrow \mathcal{F}_{G_0}$ which preserves the defining Lie supergroup triples (see [B], p.248). Taking a basis of homogeneous elements of V one interprets ρ as a holomorphic map of G_0 to matrices whose entries are superfunctions. The character of such a representation is defined by $\chi(g) := \mathrm{STr}(\rho(g))$. It is a radial superfunction on G_0 and we consider its restriction χ to a Cartan subgroup H . It is an eigenfunction of every radial operator \dot{D} . In other words there is a homomorphism λ defined on the space of radial differential operators with values in \mathbb{C} so that $\dot{D}(\chi) = \lambda(\dot{D})\chi$.

Now apply the Jacobian formula, $\dot{D}_P = JT(P)(\frac{1}{i}\frac{\partial}{\partial t})J^{-1}$, define $\tilde{\chi} = J\chi$ and observe that

$$T(P)(\frac{1}{i}\frac{\partial}{\partial t})\tilde{\chi} = \lambda(\dot{D}_P)\tilde{\chi}.$$

In other words, the eigenvalue homomorphism for the radial operator \dot{D}_P on the character χ is the same as the eigenvalue homomorphism for the constant coefficient operator $T(P)(\frac{1}{i}\frac{\partial}{\partial t})$ on the function $\tilde{\chi}$. This simple remark leads to an exact description of $T(P)$ in terms of eigenvalues of irreducible representations.

For this note that J is defined on H so that $\tilde{\chi}$ can be expanded in a Fourier series

$$\tilde{\chi}(t) = \sum_{k \in \mathbb{Z}^{m+n}} a_k e^{i\langle k, t \rangle}.$$

Applying $T(P)(\frac{1}{i} \frac{\partial}{\partial t})$ to both sides one shows that if $a_k \neq 0$, then

$$T(P)(k) = \lambda(\dot{D}_P) \quad (3)$$

for *every* extendible polynomial P . Letting P range over all such polynomials one proves the following fact.

Proposition 2.6. *The set of lattice elements k such that $a_k \neq 0$ is a W -orbit $W.k_0$.*

It should be noted that since $T(P)$ is itself W -invariant the lack of uniqueness of the lattice element k is minimal.

Now χ is W -invariant. Furthermore, for σ in the Weyl group it follows that $\sigma(J) = \varepsilon(\sigma)J$ where $\varepsilon(\sigma) = \det(\sigma) = \pm 1$. Hence, up to a multiplicative constant

$$\chi(t) = J^{-1}(t) \sum_{\sigma \in W} \varepsilon(\sigma) e^{i\langle k_0, \sigma(t) \rangle}$$

for any fixed k_0 in the support of $\tilde{\chi}$.

Now order the weight lattice so that the roots α and β which occur in the above products are positive and write

$$\chi(t) = \sum c_j e^{i\langle m_j, t \rangle}$$

where the m_j are the weights of the representation ρ with Λ being the highest weight which occurs. Compare this expression for $\chi(t)$ with that above to obtain

$$\sum c_j e^{i\langle m_j, t \rangle} \prod (e^{i\frac{\alpha(t)}{2}} - e^{-i\frac{\alpha(t)}{2}}) = \sum \varepsilon(\sigma) e^{i\langle k_0, \sigma(t) \rangle} \prod (e^{i\frac{\beta(t)}{2}} - e^{-i\frac{\beta(t)}{2}}).$$

Equating the highest order terms on each side yields

$$\Lambda + \frac{1}{2} \sum \alpha = k_0 + \frac{1}{2} \sum \beta$$

where k_0 is the highest of the elements in the W -orbit $W.k_0$. Turning this around, we see that

$$k_0 = \Lambda + \delta$$

where

$$\delta = \frac{1}{2} (\sum \alpha - \sum \beta).$$

Theorem 2.7. *Let ρ be a finite-dimensional representation of a Lie supergroup associated to one of the Lie superalgebras $\mathfrak{g} = \mathfrak{gl}(m, n), \mathfrak{osp}(2m, 2n)$. Let P be an extendible polynomial on \mathfrak{h} and $T(P)$ be the polynomial which is defined by*

$$\dot{D}_P = J^{-1}T(P)\left(\frac{1}{i}\frac{\partial}{\partial t}\right)J.$$

If χ is the character of ρ with the homomorphism λ defined by

$$\dot{D}_P = \lambda(D_P)\chi,$$

then

$$\lambda(\dot{D}_P) = T(P)(\Lambda + \delta)$$

where Λ is the highest weight of ρ .

Proof. This follows immediately from (3) and the fact that our choice of $k = k_0$ in the W -orbit is $k_0 = \Lambda + \delta$. \square

2.5 Generating functions

Here we fix \mathfrak{g} as one of the Lie superalgebras $\mathfrak{gl}(m, n)$ or $\mathfrak{osp}(2m, 2n)$ and let \dot{D}_ℓ be the radial operator defined by the particular extendible polynomial F_ℓ . Using the Fourier series development of characters of representation, it was shown above that the value $\lambda(\dot{D}_\ell)$ of the eigenvalue homomorphism on \dot{D}_ℓ for the character of an irreducible representation ρ of highest weight Λ of the associated Lie supergroup is the value of the polynomial $T(P)$ on lattice point $k_0 = \Lambda + \delta$. Letting ρ range through all such representations, we see that $T(P)$ is the unique polynomial with this property. If we think of such a point k_0 as a weight, then it is in \mathfrak{h}^* ; so we reformulate the result as follows: There is a uniquely determined polynomial function R_ℓ on \mathfrak{h}^* with $R_\ell(\Lambda + \delta) = \lambda(\dot{D}_\ell)$ on every irreducible representation of highest weight Λ .

Associated to the sequence $\{R_\ell\}$ of polynomials one has the *generating function*

$$S(z) := \sum z^\ell R_\ell(x)$$

which is computed in closed form in [B] (see Lemma 4.3, pages 327-329, for the case of $\mathfrak{gl}(m, n)$ and Lemma 4.4, pages 335-341, for $\mathfrak{osp}(2m, 2n)$). The resulting formulas for the polynomials R_ℓ are derived after the proofs of these lemmas. Using the identification $R_\ell = T(F_\ell)$, one has the following consequence which we formulate simultaneously for both $\mathfrak{gl}(m, n)$ and $\mathfrak{osp}(2m, 2n)$.

Theorem 2.8. *If \dot{D}_ℓ is the radial differential operator defined by the extendible polynomial F_ℓ with*

$$\dot{D}_\ell = J^{-1}T(F_\ell)\left(\frac{1}{i}\frac{\partial}{\partial t}\right)J, \quad (4)$$

then $T(F_\ell) \in \mathbb{C}[F_1, \dots, F_\ell]$. Moreover $T(F_\ell) = F_\ell + Q$ where $Q \in \mathbb{C}[F_1, \dots, F_{\ell-1}]$ is a polynomial of lower degree.

2.6 An application

In [CFZ] and [HPZ] characters χ of representations Lie supergroups on certain infinite-dimensional spaces play an important role. In [CFZ] the complex Lie superalgebra at hand is $\mathfrak{gl}(m, n)$ and in [HPZ] it is $\mathfrak{osp}(2n, 2n)$. In these situations one would hope to apply the above results on radial operators. However, this can not be directly done, because the characters are defined by supertrace and only converge on certain open domains \mathcal{H} in G_0 or on finite covering spaces $\widehat{\mathcal{H}}$ of such domains. On the other hand, Laplace-Casimir operators are local and can therefore be applied to such characters and in the settings of [CFZ] and [HPZ] the characters χ which appear are annihilated by Laplace-Casimir operators D_ℓ defined by the F_ℓ .

In the domains \mathcal{H} or the covering spaces $\widehat{\mathcal{H}}$ there are closed connected complex submanifolds T^+ which are either open subsets of a Cartan algebra H or lifts of such into the covering space. Now the radial operators \dot{D}_ℓ are differential operators which are apriori defined on the space \mathcal{D}_H of globally defined extendible W -invariant holomorphic functions and on that space we know how to compute them using the righthand side of (4). The restrictions of the characters χ to T^+ , which are by definition the numerical parts of $\chi|_{T^+}$, are by definition extendible as radial superfunctions, but they are only defined on T^+ and not on H . Nevertheless we wish to show that they are annihilated by the operators which are described by the righthand side of (4). For this we prove a local version of (4) and obtain the desired result on T^+ by applying the identity principle.

Local formula for \dot{D}_ℓ

We refer to a point in H as being *superregular* if it is regular in the sense of Lie theory and is not contained in any of the odd root hypersurface $\{r_\beta = 1\}$. Every a superregular point x has a basis of open neighborhoods V in \mathfrak{H} which are relatively compact in the set of superregular points in H with the property that $\sigma(V) \cap V = \emptyset$ for every $\sigma \in W \setminus \{\text{Id}\}$. Given such a V we thicken it as follows to an open neighborhood U in G_0 . Let Δ be a polydisk in \mathfrak{g}_0 which is transversal to

\mathfrak{h} and define $U = \{\exp(\xi).x; \xi \in \Delta, x \in V\}$. We choose Δ small enough so that $U \cong \Delta \times V$. For $x \in V$ fixed we think of $\exp(\Delta).x$ as a local orbit of G_0 .

Proposition 2.9. *Every superregular point x in H has a neighborhood basis of open sets V and U as above so that the restriction map $R : \mathcal{R}(U) \rightarrow \mathcal{O}(V)$ is an isomorphism.*

Proof. Since holomorphic maps $U \rightarrow \wedge \mathfrak{g}_1^*$ which are invariant by the local conjugation-action of G_0 are completely determined by their restrictions to V , it follows that R is injective. Surjectivity is proved by the following approximation argument.

First, in order to take care of W -invariance we consider the quotient $\pi : H \rightarrow Z = H/W$. The restriction $\pi|_V$ maps V biholomorphically onto a domain \tilde{V} . A basic theorem of complex analysis states that we may choose \tilde{V} (and accordingly V) so that the restriction map $\mathcal{O}(Z) \rightarrow \mathcal{O}(\tilde{V})$ has dense image.

Now let R be the product of the odd root functions on H and \tilde{R} be the associated function on Z . Define \tilde{f} be the function on \tilde{V} associated to a given holomorphic function f on V . Let \tilde{f}_n be a sequence of holomorphic functions on Z which converge to $\tilde{R}^{-2}\tilde{f}$ in $\mathcal{O}(\tilde{V})$. It follows that $\tilde{h}_n := \tilde{R}^2\tilde{f}_n$ converges to \tilde{f} . The point of this construction is that the sequence $\{h_n\}$ of lifts defined by $h_n := \pi^*(h_n)$ converge to f on V . In addition these are W -invariant and have the divisibility property. Thus we have the sequence $\{E(h_n)\}$ of radial extensions. Now the extension $h_n|_V \rightarrow E(h_n)|_V$ is such that the convergence of $h_n|_V$ implies the convergence of $E(h_n)|_V$ as a sequence of $\wedge \mathfrak{g}_1^*$ -valued holomorphic maps. Consequently the maps $E(h_n)|_U$ which are constant along the local G_0 -orbits defined by Δ also converge. If a sequence of holomorphic functions converges, then so does any induced sequence of derivatives. Thus the limit $E(f)$ of the sequence $\{E(h_n)|_U\}$ is a radial holomorphic function whose (numerical) restriction to V is the given function f . \square

Having localized to the open sets $V = U \cap H$ of superregular elements of H and proved the above extension result, given a Laplace-Casimir operator D we define its radial part on $U \cap H$ in the same way as in the global case: $\dot{D}_{U \cap H}(f) := RDE(f)$. Since the extension result was proved by taking limits of globally defined extendible functions and the global operator \dot{D} is continuous, it follows that $\dot{D}_{U \cap H}$ is just the restriction of \dot{D} to $U \cap H$. Thus we have the following local version of Theorem 2.5.

Theorem 2.10. *Under the assumptions of Theorem 2.5, let D be a Laplace-Casimir operator defined by an extendible polynomial P . If U is as above, then the*

domain of definition of the radial operator $\dot{D}_{U \cap H}$ is the full algebra of holomorphic functions $\mathcal{O}(U \cap H)$ and

$$\dot{D}_{U \cap H} = J^{-1} T(P) \left(\frac{1}{i} \frac{\partial}{\partial t} \right) J.$$

As a result we have the local version of Theorem 2.8

Theorem 2.11. *Under the assumptions of Theorem 2.8, for U as above it follows that*

$$\dot{D}_{U \cap H} = J^{-1} (\tilde{F}_\ell + Q(\tilde{F}_1, \dots, \tilde{F}_{\ell-1})) J$$

where Q is a polynomial operator of lower degree than \tilde{F}_ℓ .

Now let us return to the settings of [CFZ] and [HPZ] where we have an open piece T^+ of the Cartan algebra H contained as a closed submanifold of a domain \mathcal{H} in G_0 or a finite-to-one covering space $\hat{\mathcal{H}}$ of such a domain. The Weyl group W acts on these domains so that the above arguments apply: If $x \in T^+$ is superregular, then we setup U as above and prove the following result.

Theorem 2.12. *Let D_ℓ be the Laplace-Casimir operator defined on \mathcal{H} or $\hat{\mathcal{H}}$ by the extendible polynomial F_ℓ . Then a holomorphic superfunction on such a domain is annihilated by D_ℓ if and only if its (numerical) restriction to T^+ is annihilated by $J^{-1}(\tilde{F}_\ell + Q(\tilde{F}_1, \dots, \tilde{F}_{\ell-1}))J$.*

Proof. Since D_ℓ acts on the full sheaf of radial functions, we may regard it as acting on the restriction of given holomorphic superfunction to U . Thus its restriction to $U \cap H$ is annihilated by the associated radial operator $\dot{D}_{U \cap H}$. By Theorem 2.11 the operator $J^{-1}(F_\ell + Q(F_1, \dots, F_{\ell-1}))J$ annihilates this restriction on $U \cap H$ and the desired result follows by the identity principle. \square

Appendix: local representation of radial operators

Here, referring to [Ka1, Ka2] for details, we outline a proof of a local version of Theorem 2.5. We do this in the setting of complex Lie supergroups. Let us begin with the preparation which is required to state the result.

Recall that if G is a reductive complex Lie group with maximal complex torus H , then $h \in H$ is by definition regular if its centralizer in G is just H itself. The set of regular elements is open and dense in H and every $h_0 \in H_{reg}$ has an open neighborhood A in H such that $G.A = \{ghg^{-1}; h \in A, g \in G\}$ is G -equivariantly a product $A \times G/H$. A local version of this which can be applied, e.g., in situations where G is only acting locally on a neighborhood of h_0 in G , can be stated as

follows: There is an open neighborhood B of the identity in the submanifold of G which is defined to be the product of all H -root spaces so that $B.A$ is an open neighborhood of h_0 which is (locally) B -equivariantly the product $A \times B$.

It is shown in [Ka2] that this local product decomposition holds for Lie supergroups (G, \mathcal{F}) of type I which are equipped with a nondegenerate invariant bilinear form $b : \text{Der}(\mathcal{F}) \times \text{Der}(\mathcal{F}) \rightarrow \mathcal{F}$. Recall that “type I” means that Cartan algebras are even, i.e., are contained in \mathfrak{g}_0 and all root spaces are 1-dimensional. The complex manifolds A and B are defined exactly as above. The complex subsupermanifold structure on A is even, but nevertheless we denote it by \mathcal{A} to emphasize that it is a subsupermanifold. The subsupermanifold structure on B is not even. It is analogously denoted by \mathcal{B} .

Theorem 2.13. *For h_0 a regular element of H there is an open (supermanifold) neighborhood $\mathcal{B}.A$ which is locally \mathcal{B} -equivariantly isomorphic to the product $\mathcal{A} \times \mathcal{B}$.*

In this context a superfunction is radial if it is annihilated by all derivations in $\text{Der}(\mathcal{B})$ which, using the product structure at hand, is regarded as being contained in the space of derivations on the product neighborhood $\mathcal{A} \times \mathcal{B}$. A (super) differential operator D on this neighborhood is said to be radial if it maps radial functions to radial functions.

Using the product structure, one extends a holomorphic superfunction on \mathcal{A} , which is by definition a numerical functions, to a unique holomorphic radial superfunction on $\mathcal{A} \times \mathcal{B}$. Applying a radial operator D to such a function, we again obtain a radial function which is uniquely determined by a function on \mathcal{A} . Thus we obtain a classical differential operator $\dot{D} : \mathcal{O}(A) \rightarrow \mathcal{O}(A)$ on numerical functions.

The main source of radial operators is the center of the universal enveloping algebra. The resulting operators on $\mathcal{A} \times \mathcal{B}$ are called Laplace-Casimir operators. The local version of Theorem 2.5 which is proved in detail in [Ka1], Chapter 4, is formulated below. As in Theorem 2.5 this requires not only type I and the existence of the supersymmetric invariant form, but also that the function J is an eigenfunction of a second order Laplacian which is constructed in the proof. Thus, to simplify the formulation, we have only stated it for the cases of relevance for the applications in [HPZ] and [CFZ] where these assumptions are satisfied.

Theorem 2.14. *If \mathfrak{g} is either $\mathfrak{gl}(m, n)$ or $\mathfrak{osp}(2m, 2n)$, then for every Laplace-Casimir operator D on $\mathcal{A} \times \mathcal{B}$ there is a uniquely determined polynomial constant coefficient operator P on \mathcal{A} so that $\dot{D} = J^{-1}PJ$.*

Here J is the same globally defined function as in Theorem 2.5.

Let us conclude this appendix by commenting on the proof of this local result. As in the classical case the main point is to define a Laplace operator which in end effect is a second order Laplace-Casimir operator and whose radial part is (by an explicit computation) of the form in the theorem. Using the invariant form $b(x, y) = \text{STr}(xy)$, this operator is constructed as follows.

Fixing a splitting, one obtains a deRham complex of holomorphic differential forms in $\wedge(T_0^*M \oplus \Pi T_1^*M)$, where here M is the supermanifold $\mathcal{A} \times \mathcal{B}$ of dimension $k|\ell$. The operation Π exchanges the even and odd parts of a graded vector space. From the point of view of integration theory the correct “top-dimensional” form is a nowhere vanishing section ω_M (a “Berezinian”) of $\wedge^{k+\ell}(T_0^*M \oplus \Pi T_1^*M)$. Using b this is transformed to a top-dimensional holomorphic “volume form” ω_M in $\wedge^{k+\ell}(T_0^*M \oplus \Pi T_1^*M)$. Again using b to define contraction of ω_M with 1-forms to obtain a $*$ -operation so that $\partial * \partial f$ is a multiple $h\omega_M$ and then defining $*$ from top-dimensional forms back to functions, one obtains a Laplace operator $L_M := * \partial * \partial$ on superfunctions (see [Ka1], 4.1.1-4.1.3).

In order to define L_M one only needs a Lie supergroup of type I with a nondegenerate supersymmetric bilinear form. It is a radial operator and a formula for its radial part can be explicitly computed ([Ka1], Lemma 4.4). In order to obtain a formula of the type in Theorem 2.14 the following nontrivial result is needed. Here J is the function defined above which is qualitatively described as the square-root of the superdeterminant of the Jacobian of the mapping which identifies $\mathcal{A} \times \mathcal{B}$ with an open subset of the Lie supergroup (G, \mathcal{F}) .

Theorem 2.15. *If \mathfrak{g} is either $\mathfrak{osp}(2m, 2n)$ or $\mathfrak{gl}(m, n)$, then J is an eigenfunction of L_M .*

One way of seeing this is to first show that L_M is (up to a constant multiple) in fact the operator defined by the second order element of the center of the universal enveloping algebra which is defined in the usual way by b ([Ka1], Lemmas 4.7 and 4.8). Then one computes explicitly to show that J is indeed an eigenfunction. In general such computations are quite involved ([B] Theorems 4.1 and 4.4), but in certain cases which are relevant for applications they are simple (see [HPZ], §4.3)

Due to the fact that J is an eigenfunction one can express the radial part of the second order Laplacian as in Theorem 2.14. The general theorem follows immediately in the classical way using in addition the fact that the given Laplace-Casimir operator commutes with L_M ([Ka1], Theorem 4.1).

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